GEOGRAPHY OF BRILL-NOETHER LOCI FOR SMALL SLOPES

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Abstract

Let X be a non-singular projective curve of genus $g \geq 2$ over an algebraically closed field of characteristic zero. Let $\mathcal{M}(n,d)$ denote the moduli space of stable bundles of rank n and degree d on X and $\mathcal{W}^{k-1}_{n,d}$ the Brill-Noether loci in $\mathcal{M}(n,d)$. We prove that, if $0 \leq d \leq n$ and $\mathcal{W}^{k-1}_{n,d}$ is non-empty, then it is irreducible of the expected dimension and smooth outside $\mathcal{W}^{k}_{n,d}$. We prove further that in this range $\mathcal{W}^{k-1}_{n,d}$ is non-empty if and only if $d > 0$, $n \leq d+(n-k)g$ and $(n,d,k) \neq (n,n,n)$. We also prove irreducibility and non-emptiness for the semistable Brill-Noether loci.

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Introduction

The moduli spaces of stable vector bundles over an algebraic curve have been extensively studied from many points of view since they were first constructed more than 30 years ago, and much is now known about their detailed structure, in particular in terms of their topology. Except in the classical case of line bundles, however, relatively little is known about their geometry in terms, for example, of the existence and structure of their subvarieties.

In the case of line bundles, where the moduli spaces are all isomorphic to the Jacobian, Brill-Noether theory has long provided a basic source of geometrical information. This theory, which originated in the last century, is concerned with the subvarieties of the moduli spaces determined by bundles having at least a specified number of independent sections. Basic questions, concerning non-emptiness, connectedness, irreducibility, dimension, singularities, cohomology classes, etc., have been completely answered when the underlying curve is generic, and departures from the generic behaviour are indeed used to describe curves with special properties.

The definitions can easily be extended to bundles of any rank, but the basic questions are then far from being answered even for a generic curve. In particular, given integers n, d, k with $n \geq 2$ and $k \geq 1$, one would like to know when there exist stable (or semistable) bundles of rank n and degree d having at least k independent sections. We use the term "geography" to refer to the study of this problem (and related questions such as irreducibility and dimension of the corresponding loci) by analogy with a similar use of the term in the theory of algebraic surfaces; we shall see indeed that much of the data obtained by ourselves and others can be conveniently summarised in graphical form (see §2 and particularly Figures 1 and 2).

In order to describe some of these ideas in more detail and to state our own results, we introduce some further notation. Let X be a non-singular projective curve of genus $g \geq 2$ defined over an algebraically closed field of characteristic 0 and let $\mathcal{M}(n,d)$ denote the moduli space of stable vector bundles over X of rank n and degree d. For any integer $k \geq 1$, the *Brill-Noether locus* $\mathcal{W}^{k-1}_{n,d}$ is the set of stable bundles in $\mathcal{M}(n,d)$ having at least k independent global sections; this is in fact a subvariety of $\mathcal{M}(n,d)$ (see 1.1). (The superscript $k-1$ is used here rather than k largely for historical reasons, since projective dimension rather than vector space dimension was regarded classically as the important notion.) Associated with this locus is the number

$$\rho^{k-1}_{n,d} = n^2(g-1) + 1 - k(k - d + n(g-1)).$$

This is called the *Brill-Noether number*, and is the "expected" dimension of $\mathcal{W}^{k-1}_{n,d}$. In a similar way, we denote by $\widetilde{\mathcal{M}}(n,d)$ the moduli space of S-equivalence classes of semistable vector bundles over X and by $\widetilde{\mathcal{W}}^{k-1}_{n,d}$ the corresponding Brill-Noether locus; again this is a subvariety of $\widetilde{\mathcal{M}}(n,d)$ (see 1.2 for full definitions in this case).

In the case $n = 1$, the Brill-Noether loci (as remarked above) have been well known since the last century. In fact the variety $\mathcal{W}^{k-1}_{1,d}$ is always non-empty if $\rho^{k-1}_{1,d} \geq 0$, and connected if $\rho^{k-1}_{1,d} > 0$. The variety may be reducible, but each component has dimension at least $\rho^{k-1}_{1,d}$. For a generic curve X, $\mathcal{W}^{k-1}_{1,d}$ is empty if $\rho^{k-1}_{1,d} < 0$ and is irreducible of dimension $\rho^{k-1}_{1,d}$ with singular locus $\mathcal{W}^{k}_{1,d}$ if $g > \rho^{k-1}_{1,d} > 0$. Modern proofs of these results have been given by Kempf, Kleiman and Laksov, Fulton and Lazarsfeld, Griffiths and Harris, and Gieseker. A full treatment of this case is contained in [ACGH].

For higher rank and X generic, it is known that, for $0 < d \leq n(g-1)$, $\mathcal{W}^0_{n,d}$ is irreducible of dimension $\rho^0_{n,d}$ [Su] and $\operatorname{Sing} \mathcal{W}^0_{n,d} = \mathcal{W}^1_{n,d}$ [L]. The most extensive results to date are those of Teixidor [Te2]; these describe many cases when $\rho^{k-1}_{n,d} \geq 0$ and $\mathcal{W}^{k-1}_{n,d}$ is non-empty, as expected (see 2.5, where we shall use these results to draw our "map"). Further results on non-emptiness and irreducibility are known when $n = 2$ and $k = 2, 3$ [Su, T, Te1, Te3], while $\mathcal{W}^{k-1}_{3,1}$ and $\mathcal{W}^{k-1}_{3,2}$ are described in [NB]. On the other hand, even for X generic, $\mathcal{W}^{k-1}_{n,d}$ may have components of dimension greater than $\rho^{k-1}_{n,d}$ [BF] and the singular set of $\mathcal{W}^{k-1}_{n,d}$ may be strictly larger than $\mathcal{W}^{k}_{n,d}$ [Te2].

In this paper we consider the case when $n \geq 2$ and $0 \leq d \leq n$ and study the varieties $\mathcal{W}^{k-1}_{n,d}$ and $\widetilde{\mathcal{W}}^{k-1}_{n,d}$. Our main results, which provide a complete answer to the basic questions in this case, are:

THEOREM A : *If* $\mathcal{W}^{k-1}_{n,d}$ *is non-empty, then it is irreducible, of dimension* $\rho^{k-1}_{n,d}$ *and* $\operatorname{Sing} \mathcal{W}^{k-1}_{n,d} = \mathcal{W}^{k}_{n,d}$.

THEOREM Ã : *If* $\widetilde{\mathcal{W}}^{k-1}_{n,d}$ *is non-empty, then it is irreducible.*

THEOREM B : $\mathcal{W}^{k-1}_{n,d}$ *is non-empty if and only if*

$$d > 0, \quad n \leq d + (n-k)g \quad \text{and} \quad (n, d, k) \neq (n, n, n).$$

THEOREM B̃: $\widetilde{\mathcal{W}}^{k-1}_{n,d}$ *is non-empty if and only if either*

$$d = 0 \quad \text{and} \quad k \leq n$$

or

$$d > 0 \quad \text{and} \quad n \leq d + (n-k)g.$$

Our results give partial answers to questions 1 and 3 on the VBAC Problems List [VBAC], and are valid for all non-singular curves, not just generic ones. Note that, in the case $k \leq d < n$, Theorems B and $\widetilde{\mathrm{B}}$ follow from Teixidor's results [Te2]. Note also that the condition $n \leq d+(n-k)g$ implies that

$\rho^{k-1}_{n,d} \geq 1$, so in particular $\mathcal{W}^{k-1}_{n,d}$ is empty when $\rho^{k-1}_{n,d} = 0$ for $0 \leq d \leq n$; this gives another example where the results in higher rank differ from those in rank 1. After the work for this paper was completed, an alternative proof of Theorem $\widetilde{\text{B}}$, using variational methods based on the Yang-Mills-Higgs functional, was announced by G. Daskalopoulos and R. Wentworth [DW].

In proving our theorems, we shall distinguish the three cases $0 < d < n$, $d = 0$ and $d = n$, although all three will depend on the use of extensions of the form

$$0 \to \mathcal{O}^k \to E \to F \to 0.$$

In §1 we fix notation and give the basic definitions. In §2 we give a proof (due to G. Xiao) of Clifford's Theorem for semistable bundles (Theorem 2.1) and explain the geography of the Brill-Noether loci. In §3, we introduce the use of extensions (Proposition 3.1) and prove the necessary condition $n \leq d + (n-k)g$ in Theorems B and $\widetilde{\text{B}}$ (Theorem 3.3). In §4 we prove Theorems A and $\widetilde{\text{A}}$ when $0 < d < n$ (Theorems 4.3, 4.4). §5 provides the setting for the proofs of Theorems B and $\widetilde{\text{B}}$ which are completed in §6 (Theorem 6.3). Finally, in §§7, 8, we prove all four theorems for $d = 0$ (Theorems 7.1, 7.3) and $d = n$ (Theorems 8.2, 8.5).

Our methods yield some information on the more detailed geometry of the Brill- Noether loci (see, for example, Corollary 4.5, Theorem 7.2, Theorem 8.3). These varieties are also closely connected with various types of augmented bundle for which moduli spaces have recently been constructed. These include k-pairs [BeDW], coherent systems [LeP1, 2] (also discussed as "Brill-Noether pairs" in [KN], and just "pairs" in [Be, RV]) and extensions [BG]; for a general survey, see [BDGW]. We propose to return to these questions in future papers.

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§1 Notation and definitions

In this section, we give some basic notations and definitions.

We denote by X a non-singular projective curve of genus $g \geq 2$, fixed throughout the paper, and write $\mathcal{O}^k = \mathcal{O}_X^k$ for the trivial bundle of rank k over X. For any integers n and d with $n \geq 1$, let $\mathcal{M}(n,d)$ denote the moduli space of stable vector bundles of rank n and degree d over X. We write $\mu(E) = \deg E/\operatorname{rk} E$ for the *slope* of a bundle E.

We make no distinction between locally free sheaves and vector bundles over X. However a subsheaf of a vector bundle is called a subbundle only if the quotient is itself a vector bundle.

1.1. *Brill-Noether loci for stable bundles.* As a set of points, $\mathcal{W}_{n,d}^{k-1}$ can be defined by

$$\mathcal{W}_{n,d}^{k-1} = \{E \in \mathcal{M}(n,d) | h^0(E) \geq k\}.$$

Suppose first $(n,d) = 1$. To obtain a scheme structure on $\mathcal{W}_{n,d}^{k-1}$, let $\mathcal{U}$ be a universal bundle over $X \times \mathcal{M}(n,d)$. Choose an effective divisor D of a sufficiently large degree that $H^1(E \otimes L(D)) = 0$ for all $E \in \mathcal{M}(n,d)$. (Here

$L(D)$ is the line bundle associated to D.) Then, in the exact sequence

$$0\to H^0(E)\to H^0(E\otimes L(D))\to H^0(E|_D)\to H^1(E)\to 0,$$

the middle two terms have dimensions independent of E. Globalising this, we obtain

$$0\to \pi_*\mathcal{U}\to \pi_*(\mathcal{U}\otimes p_X^*L(D))\xrightarrow{\phi}\pi_*\left(\mathcal{U}|_{D\times\mathcal{M}(n,d)}\right)\to R^1_\pi\mathcal{U}\to 0,$$

where $\pi : X\times\mathcal{M}(n,d)\to\mathcal{M}(n,d)$ is the projection map. The middle two terms of this sequence are vector bundles.

We can now define $\mathcal{W}^{k-1}_{n,d}$ as the determinantal locus where ϕ drops rank by at least k. The "expected" dimension of $\mathcal{W}^{k-1}_{n,d}$ is given by

$$\rho^{k-1}_{n,d}=n^2(g-1)+1-k(k-d+n(g-1)),$$

which is called the *Brill-Noether number* associated to $\mathcal{W}^{k-1}_{n,d}$. It follows from the theory of determinantal varieties (see [ACGH] for further details) that, if $\mathcal{W}^{k-1}_{n,d}$ is non-empty and $\mathcal{W}^{k-1}_{n,d}\neq\mathcal{M}(n,d)$, then $\dim\mathcal{W}^{k-1}_{n,d}\geq\rho^{k-1}_{n,d}$.

For $(n,d)\neq 1$, there is no universal bundle over $X\times\mathcal{M}(n,d)$. However the above construction works for any locally universal family (for instance, over a Quot scheme); we can then define $\mathcal{W}^{k-1}_{n,d}$ to be the image of the variety so obtained under the natural morphism to $\mathcal{M}(n,d)$. It follows from geometric invariant theory that this is a closed subvariety of $\mathcal{M}(n,d)$.

1.2. *Brill-Noether loci for semistable bundles.*

Let E be a semistable bundle of rank n. Then there exists a filtration

$$0=E_0\subset E_1\subset E_2......\subset E_r=E$$

such that E_i/E_{i-1} is a stable bundle with $\mu(E_i/E_{i-1})=\mu(E)$ for $0<i\leq r$. The associated graded bundle $\bigoplus_i(E_i/E_{i-1})$ depends only on E and is denoted by $\operatorname{gr}E$.

We say that two semistable bundles are *S-equivalent* if $\mathrm{gr}\, E \cong \mathrm{gr}\, F$. There exists a moduli space $\widetilde{\mathcal{M}}(n,d)$ of S-equivalence classes of semistable vector bundles of rank n and degree d, which is an irreducible projective variety and is a natural compactification of $\mathcal{M}(n,d)$ (see [S]).

Writing $[E]$ for the S-equivalence class of E, we now define

$$\widetilde{\mathcal{W}}^{k-1}_{n,d} = \{[E] \in \mathcal{M}(n,d) | h^0(\mathrm{gr}\, E) \geq k\}.$$

Since $h^0(\mathrm{gr}\, E) \geq h^0(E)$ for all E, we can also define $\widetilde{\mathcal{W}}^{k-1}_{n,d}$ as the set of S-equivalence classes which contain a bundle E with $h^0(E) \geq k$. We can give $\widetilde{\mathcal{W}}^{k-1}_{n,d}$ a structure of variety by using a locally universal family as above. Note that this variety does not have to be the closure of $\mathcal{W}^{k-1}_{n,d}$, as there may exist components containing semi-stable bundles only. These components may have dimension smaller then $\rho^{k-1}_{n,d}$. For examples where this occurs, see §7.

1.3. *Petri map.* If $h^0(E) = k$, the tangent space to $\mathcal{W}^{k-1}_{n,d}$ at E is the kernel of the map

$$p^* : \mathrm{Ext}^1(E,E) \rightarrow H^0(E)^* \otimes H^1(E)$$

which is dual to the *Petri map*

$$p : H^0(E) \otimes H^0(E^* \otimes K) \rightarrow H^0(\mathrm{End}(E) \otimes K)$$

defined by multiplication of sections. It follows easily that $\mathcal{W}^{k-1}_{n,d}$ is smooth of dimension $\rho^{k-1}_{n,d}$ at E if and only if the Petri map is injective. (Incidentally there exist bundles E for which the Petri map is not injective [Te2, §5].)

Note also that $\mathcal{W}^{k}_{n,d} \subset \mathrm{Sing}\, \mathcal{W}^{k-1}_{n,d}$ whenever $\mathcal{W}^{k-1}_{n,d} \neq \mathcal{M}(n,d)$ (see [ACGH, Chapter II §2 and p. 189).

§2 Brill-Noether geography of vector bundles of higher ranks

Our main object in this section is to produce a "map" on which we can display the results of Brill-Noether theory for bundles of arbitrary rank. Before doing this, however, we shall state and prove a simple but fundamental result, which is a direct generalisation of Clifford's Theorem for line bundles.

Theorem 2.1 (Clifford's Theorem). *Let E be a semistable bundle of rank n and degree d with $0 \leq \mu(E) \leq 2g-2$. Then*

$$h^0(E) \leq n + \frac{d}{2}.$$

Proof: (As far as we are aware, no complete proof of this result has appeared in the literature. The following is due to G. Xiao.)

The proof is by induction on n, the case $n=1$ being the classical theorem. For E a semistable bundle of rank $n \geq 2$, note first that we can assume that $h^0(E) > 0$ and $h^1(E) > 0$ (i.e. E is *special*), for otherwise the result follows at once from Riemann-Roch.

Now let E_1 be a proper subbundle of E of maximal slope and let $E_2 = E/E_1$. Certainly E_1 and E_2 are both semistable. By semistability of E, we have $\mu(E_1) \leq 2g-2$ and $\mu(E_2) \geq 0$. On the other hand, since $h^0(E) > 0$, E possesses a subbundle of non-negative degree; so $\mu(E_1) \geq 0$. Similarly, since $h^1(E) > 0$, E possesses a quotient line bundle of degree $\leq 2g-2$; by comparing the slope of the kernel of this quotient with that of E_1, one sees easily that $\mu(E_2) \leq 2g-2$. The result now follows at once by induction. ♢

To construct our map, we first associate with $\mathcal{W}^{k-1}_{n,d}$ and $\widetilde{\mathcal{W}}^{k-1}_{n,d}$ the rational numbers

$$\lambda = \frac{k}{n}, \quad \mu = \frac{d}{n}.$$

If $d < 0$ and $k > 0$, then $\widetilde{\mathcal{W}}^{k-1}_{n,d}$ is empty, while obviously $\mathcal{W}^{k-1}_{n,d} = \mathcal{M}(n,d)$ if $k \leq 0$. We can therefore plot μ against λ in the first quadrant of the standard coordinate system (see Figure 1). Advantages of plotting things in this way are that every point with rational coordinates can in principle support bundles and that all ranks are represented in the same diagram.

In the remainder of the section, we describe some important features of the map.

2.2 *Riemann-Roch line* $\mu = \lambda + g - 1$. By the Riemann- Roch Theorem

$$h^0(E) - h^1(E) = d - n(g-1).$$

Therefore for $\mu \geq \lambda + g - 1$, i.e. above the Riemann-Roch line, $\mathcal{W}^{k-1}_{n,d}$ is the whole moduli space. Note also that any semistable bundle E with $\mu(E) > 2g - 2$ has $h^1(E) = 0$; so, for $\mu > 2g - 2$, $\widetilde{\mathcal{W}}^{k-1}_{n,d}$ is empty below the Riemann-Roch line.

2.3. *Clifford line* $\mu = 2\lambda - 2$. By Theorem 2.1 every $\mathcal{W}^{k-1}_{n,d}$ below this line is empty.

The interesting part of the map is therefore the pentagonal region bounded by the axes, the Riemann-Roch line, the Clifford line and the line $\mu = 2g - 2$. This corresponds to the region in which there may exist special semistable bundles.

2.4. *Brill-Noether curve.* Define

$$\tilde{\rho} = \frac{1}{n^2}(\rho^{k-1}_{n,d} - 1) = (g-1) - \lambda(\lambda - \mu + (g-1)).$$

We call the curve $\tilde{\rho} = 0$ the *Brill-Noether curve*. The curve is a branch of a hyperbola, below which the expectation is that the Brill-Noether loci will be finite.

2.5. *Teixidor parallelograms.* In [Te2], Teixidor defines ranges of values for n, d, k such that for generic curves the $\mathcal{W}^{k-1}_{n,d}$ are non-empty and have a component of the expected dimension. These ranges correspond to points (λ, μ) lying in or on one of the parallelograms marked T on the map. These parallelograms have vertices at integer points, sides parallel to $\lambda = 0$ and $\mu = \lambda$ and have all their vertices on or above the Brill-Noether curve 2.4. If all the vertices lie above $\tilde{\rho} = 0$, then $\mathcal{W}^{k-1}_{n,d}$ is non-empty whenever (λ, μ) lies in or on the parallelogram. If the lower right vertex of the parallelogram lies on $\tilde{\rho} = 0$, this still holds with the possible exception of those points of the parallelogram with the same μ-coordinate as this vertex; for such points, Teixidor shows only that $\widetilde{\mathcal{W}}^{k-1}_{n,d}$ is non-empty.

FIGURE 1

In this paper we are concerned with the region $0 \leq \mu \leq 1$ of the map (see Figure 2). The subregion $\lambda \leq \mu < 1$ lies in a Teixidor parallelogram, but the remainder of the region does not. In any case, Teixidor proves only non-emptiness (and, for X generic, the existence of a component of the correct dimension), whereas we shall solve the non-emptiness, irreducibility and singularity problems for the entire region.

A key rôle in this is played by the tangent line at $(1, 1)$ to $\tilde{\rho} = 0$. This is given by $\mu + (1 - \lambda)g = 1$ or equivalently $n = d + (n - k)g$. Thus the inequality $n \leq d + (n - k)g$ in Theorems B and $\widetilde{\text{B}}$ describes the area on or above this tangent line. Theorem B therefore states that for $\mu \leq 1$, $\mathcal{W}^{k-1}_{n,d}$ is empty below this line, while Theorem $\widetilde{\text{B}}$ says that the same is true for $\widetilde{\mathcal{W}}^{k-1}_{n,d}$

except on $\mu = 0$. On the other hand, the Brill-Noether number $\rho_{n,d}^{k-1}$ can be positive below the line, so this is not a sufficient condition for the non-emptiness of $\mathcal{W}_{n,d}^{k-1}$. This phenomenon can be compared with the "fractal mountain range" of Drezet and Le Potier, which excludes the existence of some stable bundles on $\mathbf{P}^2$, which should exist for purely dimensional reasons [DL].

FIGURE 2

§3 Emptiness of Brill-Noether loci

In this section, we assume that E has rank $n \geq 2$ and that either E is stable and $\mu(E) \leq 1$ or E is semistable and $\mu(E) < 1$. Our main purpose is to prove the necessity of the conditions in Theorems B and $\widetilde{\mathrm{B}}$ (see Theorem 3.3).

We begin with the following proposition, which will be used many times in the paper.

Proposition 3.1. *Let E be a stable bundle of degree d, $0 \leq d \leq n$ (or a semistable bundle with $0 \leq d < n$), and $h^0(E) \geq k > 0$. Let V be a subbundle of E generated by k independent global sections of E. Then V is a trivial bundle of rank k.*

Proof: We have the exact sequence

$$0 \to V \to E \to F \to 0.$$

If V is non-trivial, there exists a section $s \in h^0(E)$ such that $\deg D > 0$, where D is the divisor of zeros of s. Then $\deg L(D) > 0$ and $\mu(L(D)) \geq 1$.

But $L(D)$ is a subbundle of a stable (resp. semistable) bundle E and $\mu(E) \leq 1 (resp. < 1)$. This is a contradiction, so $V \cong \mathcal{O}^k$. $\diamondsuit$

Remark 3.2. i) The above implies that for any $E \in \mathcal{W}_{n,d}^{k-1}$, $0 \leq d \leq n$, E can be presented as an extension of the form

$$0 \to \mathcal{O}^k \to E \to F \to 0. \tag{1}$$

Similarly, every point of $\widetilde{\mathcal{W}}_{n,d}^{k-1}$, $0 \leq d < n$, has a representative E which can be presented in this form.

ii) Note that, if $d \geq 0$ and E is stable, or $d > 0$ and E is semistable, then $h^0(E^*) = 0$. Except in the case $d = 0$, E semistable, we may therefore assume that $h^0(F^*) = 0$ in the above sequence, as F^* is a subbundle of E^*.

Theorem 3.3. *$\mathcal{W}_{n,d}^{k-1}$ is empty for $d > 0$, $n > d + (n-k)g$ and for $d = 0$. $\widetilde{\mathcal{W}}_{n,d}^{k-1}$ is empty for $d > 0$, $n > d + (n-k)g$ and for $d = 0$, $k > n$.*

Theorem 3.3 has also been proved in the case $d > 0$ by Anne Maisani.

Proof.: By Remark 3.2(i), every point of $\widetilde{\mathcal{W}}_{n,d}^{k-1}$ can be represented by a bundle E of the form (1). It follows at once that $\widetilde{\mathcal{W}}_{n,d}^{k-1}$ is empty if $k > n$ or if $k = n$ and $d > 0$. Moreover, if $d = 0$, (1) contradicts the stability of E; so $\mathcal{W}_{n,d}^{k-1}$ is empty. We can therefore suppose that $k < n$ and $d > 0$.

In this case, the extensions (1) are classified by the elements of the vector space $H = \bigoplus^k H^1(F^*)$, i.e. by k-tuples $(e_1, \ldots, e_k)$ with $e_i \in H^1(F^*)$. Moreover two extensions are isomorphic if the corresponding points are in the same orbit of the natural action of $GL(k)$ on H. Thus, if $e_1, \ldots, e_k$ are linearly dependent, we can suppose (using this action) that $e_k = 0$; hence the extension has a partial splitting to give $\mathcal{O}$ as a direct summand of E, contradicting the stability hypothesis.

Now, since $h^0(F^*) = 0$ by Remark 3.2(ii), we have

$$h^1(F^*) = d + (n-k)(g-1).$$

So $e_1, \ldots, e_k$ are necessarily linearly dependent if $k > d + (n-k)(g-1)$, or equivalently $n > d + (n-k)g$. $\diamondsuit$

For future convenience we finish this section with the following proposition.

Proposition 3.4. *Let F be a fixed bundle of rank $n-k$ and degree d with $h^0(F^*) = 0$. Then, if $n \leq d + (n-k)g$, the extensions*

$$0 \to \mathcal{O}^k \to E \to F \to 0$$

with no trivial summands are classified up to automorphism of $\mathcal{O}^k$ by a variety of dimension $k(d + (n-k)g - n)$.

Proof : The extensions of this form are classified by the linearly independent k-tuples of elements of $H^1(F^*)$ modulo the linear action of $GL(k)$, in other words by the Grassmannian $Grass_k(H^1(F^*))$. Now

$$\begin{aligned} \dim Grass_k(H^1(F^*)) = k(h^1(F^*) - k) &= k(d + (n-k)(g-1) - k) \\ &= k(d + (n-k)g - n). \end{aligned}$$

$\diamondsuit$

§4 Irreducibility

In this section we shall use Proposition 3.1 and Remark 3.2 to prove Theorems A and Ã when $0 < d < n$. We begin with a lemma which is probably well known (and certainly frequently assumed), but which we could not find in a suitable form in the literature (see [Ty, Theorem 2.5.1] for a similar result).

Let $\mathcal{F}$ be a bounded set of non-stable bundles of rank n and degree d. Then there exists a finite number of families of bundles of rank n over X, parametrised by varieties V_α, including representatives of all bundles in the given set (up to isomorphism). For $v \in V_\alpha$, let E_v denote the corresponding bundle over X, and let $n_{\alpha,v}$ denote the dimension of the closure of the set $\{w \in V_\alpha | E_w \cong E_v\}$ in V_α. Write

$$m_\alpha = \min\{n_{\alpha,v} | v \in V_\alpha\}, \quad p = \max_\alpha\{\dim V_\alpha - m_\alpha\}.$$

In these circumstances, we shall say that $\mathcal{F}$ *depends on at most* p *parameters.*

Lemma 4.1. *Any bounded set* $\mathcal{F}$ *of non-stable bundles of rank* n *depends on at most* $n^2(g-1)$ *parameters.*

Remark 4.2. i) Since stable bundles of rank n and degree d depend on precisely $n^2(g-1)+1$ parameters, this means that for counting problems we can assume that the dimension of any bounded family of vector bundles of rank n is at most $n^2(g-1)+1$.

ii) If $g = 1$, Lemma 4.1 is not true. Actually there are "more" unstable than stable bundles in this case (see [A]).

Proof of Lemma 4.1: If E is a non-stable vector bundle of rank n then there exists a filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset ... \subset E_r = E$$

with E_i/E_{i-1} stable and $\mu(E_i/E_{i-1}) \leq \mu(E_{i-1}/E_{i-2})$. For $E \in \mathcal{F}$, the ranks and degrees of the E_i can take only finitely many values, so we can suppose these ranks and degrees are all fixed.

Let

$$\mathrm{rk}\,(E_i/E_{i-1}) = n_i, \qquad \deg(Hom(E_j/E_{j-1}, E_i/E_{i-1})) = d_{j,i}$$

and let β_i be the minimum number of parameters on which the set of bundles which can occur as E_i in the above filtration depends. From the exact sequence

$$0 \to E_1 \to E_2 \to E_2/E_1 \to 0$$

we have that

$$\beta_2 \leq n_1^2(g-1) + 1 + n_2^2(g-1) + 1$$

if $h^1(Hom(E_2/E_1, E_1)) = 0$ for all E_1, E_2/E_1 and

$$\beta_2 \leq n_1^2(g-1) + 1 + n_2^2(g-1) + 1 + \max\{h^1(Hom(E_2/E_1, E_1))\} - 1$$

otherwise, since E_1 and E_2/E_1 are stable. In the first case, clearly

$$\beta_2 \leq (n_1 + n_2)^2(g-1).$$

In the second, since $Hom(E_2/E_1, E_1)$ is semistable and $d_{2,1} \geq 0$, we have by Clifford's theorem

$$h^0(Hom(E_2/E_1, E_1)) \leq \frac{d_{2,1}}{2} + n_1 n_2. \tag{2}$$

So by Riemann-Roch $h^1(Hom(E_2/E_1, E_1)) \leq n_1 n_2 g - \frac{d_{2,1}}{2}$.

Therefore

$$\begin{aligned} \beta_2 &\leq (n_1^2 + n_2^2)(g-1) + 1 + n_1 n_2 g - \tfrac{d_{2,1}}{2} \\ &= (n_1 + n_2)^2(g-1) + 1 - n_1 n_2 (g-2) - \tfrac{d_{2,1}}{2} \\ &\leq (n_1 + n_2)^2(g-1) \end{aligned}$$

unless $g = 2$ and $d_{2,1} = 0$. In the exceptional case, the left-hand side of (2) is 0 unless $E_2/E_1 \cong E_1$, when it is 1; so the inequality can be improved unless E_1 and E_2/E_1 are isomorphic line bundles. But the extensions of the required form in which E_1 and E_2/E_1 are isomorphic line bundles depend on

$$2g-1 \leq 4(g-1)$$

parameters. This completes the proof for $r=2$.

For $r \geq 3$, we proceed by induction on r. The same argument as above gives

$$\beta_r \leq \beta_{r-1} + n_r^2(g-1) + 1 + \max\{h^1(Hom(E_r/E_{r-1}, E_{r-1}))\} - 1$$

(unless $h^1(Hom(E_r/E_{r-1}, E_{r-1}))$ is always zero, in which case there is a better estimate as above). Now

$$\begin{aligned} h^1(Hom(E_r/E_{r-1}, E_{r-1})) &\leq \sum_{i=1}^{r-1} h^1(Hom(E_r/E_{r-1}, E_i/E_{i-1})) \\ &\leq \sum_{i=1}^{r-1}(n_i n_r g - \frac{d_{r,i}}{2}) \end{aligned}$$

by Clifford's Theorem and Riemann-Roch. So

$$\beta_r \leq \beta_{r-1} + n_r^2(g-1) + 2\sum_{i=1}^{r-1} n_i n_r (g-1),$$

and the result follows from the inductive hypothesis. ♢

We are now ready to prove Theorem Ã when $0 < d < n$.

Theorem 4.3. *If $0 < d < n$ and $\widetilde{\mathcal{W}}_{n,d}^{k-1}$ is non-empty, then it is irreducible.*

Proof: By Proposition 3.1 and Remark 3.2, any point of $\widetilde{\mathcal{W}}_{n,d}^{k-1}$ has a representative E of the form (1) with $h^0(F^*) = 0$. For fixed rank and degree, the set $\{F | h^0(F^*) = 0\}$ is bounded. It follows by a standard argument (due originally to Serre, see for example [A, Theorem 2]) that there is an irreducible family which includes representatives of all such bundles. The condition $h^0(F^*) = 0$ defines an open subfamily, parametrised by an irreducible variety Y. The required extensions are then parametrised by a projective bundle over Y, and those for which E is semistable by an open subset of the total space of this bundle. This subset is again irreducible and maps onto $\widetilde{\mathcal{W}}^{k-1}_{n,d}$. So $\widetilde{\mathcal{W}}^{k-1}_{n,d}$ is irreducible. $\diamondsuit$

Next we prove Theorem A for $0 < d < n$.

Theorem 4.4. *If* $0 < d < n$ *and* $\mathcal{W}^{k-1}_{n,d}$ *is non-empty, then it is irreducible of dimension* $\rho^{k-1}_{n,d}$. *Moreover* $\operatorname{Sing} \mathcal{W}^{k-1}_{n,d} = \mathcal{W}^{k}_{n,d}$.

Proof : Suppose $\mathcal{W}^{k-1}_{n,d}$ is not empty. Since $\mathcal{W}^{k-1}_{n,d}$ is an open subset of $\widetilde{\mathcal{W}}^{k-1}_{n,d}$, it is irreducible. From 1.1 we know that $\rho^{k-1}_{n,d} \leq \dim \mathcal{W}^{k-1}_{n,d}$.

Given n, d, k, let $\mathcal{S}$ be the set of all possible extensions

$$0 \to \mathcal{O}^k \to E \to F \to 0$$

with $h^0(F^*) = 0$, $\operatorname{rk} F = n - k$ and $\deg F = d$, such that E does not have trivial summands. From Remark 4.2(i) and Propositions 3.1 and 3.4 we obtain

$$\begin{aligned} \dim \mathcal{W}^{k-1}_{n,d} &\leq \text{the number of parameters on which } \mathcal{S} \text{ depends} \\ &\leq (n-k)^2(g-1) + 1 + k(d + (n-k)g - n) \\ &= \rho^{k-1}_{n,d} \end{aligned}$$

Therefore, $\dim \mathcal{W}^{k-1}_{n,d} = \rho^{k-1}_{n,d}$.

To see that $\operatorname{Sing} \mathcal{W}^{k-1}_{n,d} = \mathcal{W}^{k}_{n,d}$, note first that, since $\rho^{k-1}_{n,d} > \rho^{k}_{n,d}$,

$$\mathcal{W}^{k-1}_{n,d} \neq \mathcal{W}^{k}_{n,d}.$$

Now let $E \in \mathcal{W}^{k-1}_{n,d} - \mathcal{W}^{k}_{n,d}$, so that $H^0(E) \cong H^0(\mathcal{O}^k)$. Since

$$\begin{aligned} H^0(E) \otimes H^0(E^* \otimes K) &\cong H^0(\mathcal{O}^k) \otimes H^0(E^* \otimes K) \\ &\cong H^0(\mathcal{O}^k \otimes E^* \otimes K) \\ &\hookrightarrow H^0(E \otimes E^* \otimes K), \end{aligned}$$

the Petri map is injective. So, by 1.3, $\mathcal{W}^{k-1}_{n,d}$ is smooth at E. Since $\mathcal{W}^{k}_{n,d} \subset \operatorname{Sing} \mathcal{W}^{k-1}_{n,d}$ by 1.3, we have $\operatorname{Sing} \mathcal{W}^{k-1}_{n,d} = \mathcal{W}^{k}_{n,d}$ as required. ♢

Theorems 4.3 and 4.4 complete the proof of Theorems A and Ã when $0 < d < n$. The cases $d = 0$ and $d = n$ will be covered in §§7, 8.

We finish this section with

Corollary 4.5. *If $(n-k, d) = 1$ and $\mathcal{W}^{k-1}_{n,d}$ is non-empty, then there is a dominant rational map $g : Grass_k(\mathcal{R}^1_p(\mathcal{U}^*)) - - \rightarrow \mathcal{W}^{k-1}_{n,d}$, where $\mathcal{U}$ is the universal bundle over $X \times \mathcal{M}(n-k, d)$ and p the projection to $\mathcal{M}(n-k, d)$.*

Proof : By Lemma 4.1 and the proof of Theorem 4.4, the stable bundles E constructed from non-stable F belong to a proper subvariety of $\mathcal{W}^{k-1}_{n,d}$. The corollary now follows from the proofs of Proposition 3.4 and Theorem 4.4. ♢

§5 A criterion for non-emptiness

In this section we will give the setting that we need to prove Theorems B and B̃ for $0 < d < n$. More precisely, we shall give a criterion for the non-emptiness of $\mathcal{W}^{k-1}_{n,d}$ by estimating the number of conditions on an extension (1) which are required for E to be non-stable.

Assume that $0 < d < n$ and let F be a stable bundle of rank i and degree d. Let

$$\xi : 0 \to \mathcal{O}^k \to E \to F \to 0$$

be an extension of F by $\mathcal{O}^k$ such that E does not have trivial summands. By the proof of Theorem 3.3, such ξ exist if and only if $k + i = n \leq d + ig$.

If E is non-stable, then it has a stable quotient bundle H of rank $s < n$ and degree d' such that

$$\mu(H) \leq \mu(E). \tag{3}$$

This fits in the following diagram:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}^k & \to & E & \to & F & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & H & \to & H_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{4}$$

where M is the image of $\mathcal{O}^k \to E \to H$. Note that $M \neq 0$; otherwise there would exist a non-zero homomorphism $g : F \to H$. Since both are stable, $\mu(F) \leq \mu(H)$. Since $\mu(E) < \mu(F)$, this contradicts (3). Moreover, since M has non-negative degree and H is stable, $\deg H \geq 0$

Since M is generated by its global sections, it must be trivial. Otherwise there would exist a section of H generating a line bundle of positive degree; in conjunction with (3), this contradicts the stability of H. For the same reason, H_1 must be torsion-free (and hence locally free).

One can now complete diagram (4) as follows

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}^{n-s-l} & \to & G & \to & G_1^l & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}^{k} & \to & E^n & \to & F^i & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}^{s-m} & \to & H^s & \to & H_1^m & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\qquad (5)
$$

(where the superscripts denote the ranks of the various bundles). Note that $m > 0$; otherwise H would be trivial and E would have a trivial summand. Also $l > 0$; otherwise

$$\deg H = \deg H_1 = \deg F = d,$$

contradicting (3).

Note that the existence of the top sequence in (5) implies that the k-tuple of elements of $H^1(F^*)$ defining ξ maps under the surjective homomorphism $H^1(F^*) \to H^1(G_1)$ to a k-tuple of which at most $(n-s-l)$ components are linearly independent. Since, by Riemann-Roch,

$$h^1(G_1^*) \geq d - d' + l(g-1),$$

this rank condition defines a subvariety Z in $\bigoplus^k H^1(F^*)$ with

$$\operatorname{codim} Z \geq (s-m)(d-d'+lg-n+s). \qquad (6)$$

On the other hand, the stability of F implies that every quotient bundle of H_1 has slope greater than every subbundle of G_1, and hence that $h^0(H_1^* \otimes G_1) = 0$. So

$$h^1(H_1^* \otimes G_1) = ld' - m(d-d') + lm(g-1).$$

Since F varies in a bounded set, so do G_1 and H_1 (note that $d - d' > 0$ by (3)); so, by Lemma 4.1, the non-trivial extensions occuring in the right-hand column of (5) depend on at most

$$l^2(g-1) + 1 + m^2(g-1) + 1 + ld' - m(d-d') + lm(g-1) - 1$$
$$= \dim \mathcal{M}(i,d) + ld' - m(d-d') - lm(g-1) \qquad (7)$$

parameters.

Proposition 5.1. *Let* $0 < d < n$. *If*

$$(s-m)(d-d'+lg-n+s) > ld' - m(d-d') - lm(g-1)$$

for all possible choices of s, d', m *and* l, *then* $\mathcal{W}_{n,d}^{k-1}$ *is non-empty.*

Proof: By (7), the general element F of $\mathcal{M}(i,d)$ admits families of extensions $0 \to G_1 \to F \to H_1 \to 0$ as above depending on at most

$$ld' - m(d-d') - lm(g-1)$$

parameters. If the inequality holds, it follows from (6) that there exists a non-empty open set of extensions ξ for which no diagram (5) exists. If this holds for all possible choices of s, d', m and l (of which there are finitely many), then the general extension ξ must define a stable bundle E. $\diamondsuit$

We will use Proposition 5.1 to prove Theorems B and $\tilde{\mathrm{B}}$ for $0 < d < n$. In view of Theorem 3.3, it is sufficient to show that the inequality of Proposition 5.1 holds whenever the numerical conditions needed for (5) to exist hold. For convenience, we restate these conditions now.

In the first place, (3) can be stated as

$$sd - nd' \geq 0. \qquad (a)$$

The stability of F implies that

$$(l+m)d' - md > 0. \qquad (b)$$

Since H is stable, we have from the proof of Theorem 3.3

$$d' - s + mg \geq 0. \qquad (c)$$

Finally the inequality in Proposition 5.1 can be written as

$$m(n-s-l) - d'(l+s) + s(d+lg-n+s) > 0. \qquad (d)$$

In the next section, we shall prove that (a), (b) and (c) imply (d), thus completing the proofs.

Remark 5.2. The necessary condition

$$n \leq d + (n-k)g$$

of Theorems B and $\tilde{\mathrm{B}}$ does not enter the calculation explicitly. In fact this inequality is a consequence of the hypotheses of Proposition 6.1 below.

§6 Proof of the inequality

Our object in this section is to prove

Proposition 6.1. *Suppose* (a), (b) *and* (c) *hold with*

$$0 < d < n, \quad 0 < s \leq n-l \quad \text{and} \quad l > 0.$$

Then (d) *holds.*

It will be helpful for the proof to represent some of the data in a geometrical form. We do this as follows:

FIGURE 3

In this figure, we regard n, d, s and l as fixed and m, d' as variables. The curve q is (a branch of) the hyperbola

$$(l+m)d' - md = 0$$

defining the inequality (b) and has the form indicated since $l > 0$. The lines ℓ_a and ℓ_c defining the inequalities (a) and (c) depend on s, but the line ℓ joining C (the intersection of ℓ_a and ℓ_c) to $(0,0)$ has equation

$$(n-d)d' = dmg$$

which is independent of s. The shaded region is the region where $(a), (b)$ and (c) are all satisfied.

Lemma 6.2. *Suppose the hypotheses of Proposition 6.1 hold. Then*

$$(n-d)s \geq n(n-d-lg). \qquad (*)$$

Proof: Note that, for any s, the line ℓ_c has slope $-g < 0$. It follows that, if $(a), (b)$ and (c) hold, then the point C must lie above q.

Now ℓ meets q at $(0,0)$ and the point D with coordinates

$$m = \frac{n-d-lg}{g}, \qquad d' = \frac{d(n-d-lg)}{n-d}.$$

Since ℓ has positive slope, the m-coordinate of C must be at least as great as that of D, i.e.

$$\frac{(n-d)s}{ng} \geq \frac{n-d-lg}{g}.$$

Clearing denominators, this gives $(*)$. (Note that the coordinates of D in this proof could be negative; this does not affect the argument.) $\diamondsuit$

Proof of Proposition 6.1: The value of the LHS of (d) at C is

$$\frac{(n-d)s}{ng}(n-s-l)-\frac{ds}{n}(l+s)+s(d+lg-n+s).$$

A simple calculation shows that this is equal to

$$\frac{s(g-1)}{ng}[s(n-d)-n(n-d-lg)+l(n-d)].$$

It follows at once from Lemma 6.2 that this is positive. In other words, C lies below the line defining the inequality (d), which has non-negative slope. So the whole region in which (a), (b) and (c) all hold also lies below this line.

$\diamondsuit$

We are now ready to state

Theorem 6.3. *Theorems* B *and* $\widetilde{\mathrm{B}}$ *hold for* $0<d<n$.

Proof: This follows from Theorem 3.3 and Propositions 5.1 and 6.1. $\diamondsuit$

§7 The case $\mu=0$

Theorem 7.1. *Theorems* A *and* B *hold for* $d=0$.

Proof: For bundles of degree 0, the existence of a section contradicts stability; so $\mathcal{W}^{k-1}_{n,d}$ is always empty. This gives Theorem B, and Theorem A holds trivially. $\diamondsuit$

On the other hand, we have

Theorem 7.2. *For $1 \leq k \leq n$, there exists a bijective morphism*

$$\widetilde{\mathcal{M}}(n-k,0) \to \widetilde{\mathcal{W}}^{k-1}_{n,0}.$$

Proof: If E is a semistable bundle of degree 0 with k independent sections, then by Proposition 3.1 we have an extension

$$0 \to \mathcal{O}^k \to E \to F \to 0.$$

So E is S-equivalent to $\mathcal{O}^k \oplus F$ for some semistable bundle F of rank $n-k$ and degree 0.

Hence the formula $[F] \to [\mathcal{O}^k \oplus F]$ defines a bijection from $\widetilde{\mathcal{M}}(n-k,0)$ to $\widetilde{\mathcal{W}}^{k-1}_{n,0}$. Since $\widetilde{\mathcal{M}}(n-k,0)$ is a coarse moduli space, this is a morphism. $\diamondsuit$

Theorem 7.3. *Theorems* $\widetilde{\mathrm{A}}$ *and* $\widetilde{\mathrm{B}}$ *hold for* $d=0$.

Proof: By Theorem 3.3, $\widetilde{\mathcal{W}}^{k-1}_{n,0}$ is empty if $k > n$. The rest of Theorem $\widetilde{\mathrm{B}}$ now follows from Theorem 7.2, as does Theorem $\widetilde{\mathrm{A}}$ when we recall that $\widetilde{\mathcal{M}}(n-k,0)$ is irreducible. $\diamondsuit$

Remark 7.4. Note that

$$\dim \widetilde{\mathcal{M}}(n-k,0) = (n-k)^2(g-1)+1 < \rho^{k-1}_{n,0}$$

if $n < (n-k)g$. This is no contradiction since the points of $\widetilde{\mathcal{W}}^{k-1}_{n,0}$ correspond to S-equivalence classes of bundles, not isomorphism classes.

§8 The case $\mu = 1$

In this final section we prove our theorems for the case $d = n$.

For stable bundles the key result is

Proposition 8.1. *$\mathcal{W}_{n,n}^{n-2}$ is non-empty.*

Proof: Consider the extensions

$$0 \to \mathcal{O}^{n-1} \to E \to F \to 0,$$

where F is a line bundle of degree n. Since $n \le n+g$, there exist extensions of this form for which E has no trivial summands. If E is non-stable, we have as in §5 a diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}^{n-1} & \to & E & \to & F & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & H & \to & H_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

with H stable and $\mu(H) \le \mu(E)$. If H_1 is a line bundle, then $H_1 \cong F$; so

$$\deg H = \deg F + \deg M \ge d$$

and $\mu(H) > \mu(E)$, which is a contradiction.

It follows that H_1 must be a torsion sheaf. If $\mu(H) < 1$, this contradicts the stability of H just as in §5. However, if $\mu(H) = 1$, it is possible for H to have a section with a zero. This can happen only if $H = \mathcal{O}(x)$ for some $x \in X$. Moreover, in this case, we cannot have $M = \mathcal{O}(x)$, since $\mathcal{O}(x)$ is not generated by global sections, so our diagram must become

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}^{n-2} & \to & G & \to & F(-x) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}^{n-1} & \to & E & \to & F & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O} & \to & \mathcal{O}(x) & \to & \mathcal{O}_x & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The existence of this diagram implies that the $(n-1)$-tuple of elements of $H^1(F^*)$ corresponding to the extension

$$0 \to \mathcal{O}^{n-1} \to E \to F \to 0$$

must become dependent in $H^1(F(-x)^*)$. Now

$$h^1(F(-x)^*) = n + g - 2 \geq n - 1;$$

so this condition defines a subvariety of $\bigoplus^{n-1} H^1(F^*)$ of codimension $g > 1$. Since $F(-x)$ depends on only one parameter, we can find an extension for which no such diagram exists. $\diamondsuit$

Theorem 8.2. *Theorems* A *and* B *hold for* $d = n$.

Proof: Proposition 3.1 remains true for stable bundles when $d = n$. The arguments of §4 therefore apply to prove Theorem A in this case.

For Theorem B, it follows from Proposition 8.1 that $\mathcal{W}_{n,n}^{k-1}$ is non- empty for $k \leq n-1$. On the other hand $\mathcal{W}_{n,n}^{n-1}$ is certainly empty, since a bundle with n independent sections is either trivial or has a section with a zero;

when $d = n$, either possibility contradicts stability. Thus Theorem B holds when $d = n$. $\diamondsuit$

We now turn to the semistable case. As in the case $d = 0$, we obtain a result which is interesting in its own right.

Theorem 8.3. *Let S^nX denote the nth symmetric power of X. Then there exists a bijective morphism $S^nX \to \widetilde{\mathcal{W}}^{n-1}_{n,n}$.*

Proof: Let E be a semistable bundle of rank and degree n with n independent sections. Since $E \not\cong \mathcal{O}^n$, it must possess a section with a zero. Semistability then gives an extension

$$0 \to \mathcal{O}(x) \to E \to E' \to 0,$$

where E' is semistable of rank and degree $n-1$ and has $n-1$ independent sections. It follows by induction that E is S-equivalent to a bundle of the form $\mathcal{O}(x_1) \oplus \ldots \oplus \mathcal{O}(x_n)$. The existence of the required morphism follows from the universal properties of S^nX and $\mathcal{M}(n,n)$. $\diamondsuit$

We need also

Proposition 8.4. *For $k < n$, the point $[E] \in \widetilde{\mathcal{W}}^{k-1}_{n,n}$ determined by a semistable bundle E lies in the closure of $\mathcal{W}^{k-1}_{n,n}$.*

Proof: For those bundles E which can be expressed as extensions

$$0 \to \mathcal{O}^k \to E \to F \to 0,$$

we argue exactly as in Theorem 4.3.

The remaining bundles are those which possess a section with a zero. We then have an extension

$$0 \to \mathcal{O}(x) \to E \to F \to 0,$$

so that E is S-equivalent to $\mathcal{O}(x) \oplus F$. We can suppose inductively that $[F]$ belongs to the closure of $\mathcal{W}^{k-2}_{n-1,n-1}$. (Note that, in the case $n = 2$, F is a line bundle, so this is trivial. More generally, it is trivial whenever $k = 1$, since then $\mathcal{W}^{k-2}_{n-1,n-1} = \mathcal{M}(n-1, n-1)$.) It is therefore sufficient to prove the proposition when $E \cong \mathcal{O}(x) \oplus F$ and $F \in \mathcal{W}^{k-2}_{n-1,n-1}$.

For this, we consider extensions

$$0 \to F \to E' \to \mathcal{O}(x) \to 0,$$

Note that we have an inclusion $\mathcal{O} \subset \mathcal{O}(x)$ and that this section of $\mathcal{O}(x)$ lifts to E' if and only if the pull-back of the extension by this inclusion is trivial. We therefore have a family of such extensions parametrised by

$$V = Ker[H^1(\mathcal{O}(x)^* \otimes F) \to H^1(F)].$$

Now, since F is stable with $\mu(F) = 1$,

$$h^1(\mathcal{O}(x)^* \otimes F) = (n-1)(g-1)$$

and

$$h^1(F) = h^0(F) - (n-1) + (n-1)(g-1) < (n-1)(g-1).$$

So $\dim V \geq 1$ and there exist non-trivial extensions

$$0 \to F \to E' \to \mathcal{O}(x) \to 0$$

with $h^0(E') \geq k$.

Now suppose E' is such an extension, and that it possesses a section with a zero. Since F is stable, this cannot be a section of F, so it maps to a section of $\mathcal{O}(x)$. The corresponding subbundle must map isomorphically to $\mathcal{O}(x)$, splitting the extension.

It follows that V parametrises a family such that the general member has a subbundle $\mathcal{O}^k$ and therefore defines a point in the closure of $\mathcal{W}_{n,n}^{k-1}$, while the special member corresponding to $0 \in V$ is $\mathcal{O}(x) \oplus F$. Hence $[\mathcal{O}(x) \oplus F]$ is in the closure of $\mathcal{W}_{n,n}^{k-1}$ as required. $\diamondsuit$

We now have finally

Theorem 8.5. *Theorems* $\widetilde{\mathrm{A}}$ *and* $\widetilde{\mathrm{B}}$ *hold for* $d = n$.

Proof: $\widetilde{\mathcal{W}}_{n,n}^{k-1}$ is irreducible for $k = n$ by Theorem 8.3 and for $k < n$ by Theorem 8.2 and Proposition 8.4. On the other hand, the proof of Theorem 8.3 shows that no semistable bundle with $d = n$ can have more than n independent sections. $\diamondsuit$

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